Output Feedback Boundary Control of a Heat PDE Sandwiched Between Two ODEs

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***Abstract*—We present designs for exponential stabilization of an ordinary differential equation (ODE)-heat partial differential equation (PDE)-ODE coupled system where the control actuation only acts in one ODE. The combination of PDE backstepping and ODE backstepping is employed in a state feedback control law and in an observer that estimates PDE and two ODE states using only one PDE boundary measurement. Based on the state feedback control law and the observer, the output feedback control law is then proposed. The exponential stability of the closed-loop system and the boundedness and exponential convergence of the control law are proved via Lyapunov analysis. Finally, numerical simulations validate the effectiveness of this method for the "sandwiched" system.**

***Index Terms*—Backstepping, distributed parameter systems, ODE-PDE-ODE, parabolic systems.**

I. Introduction

A. Control of Parabolic Partial Differential Equations (PDEs)

Parabolic PDEs are predominately used in describing fluid, thermal, and chemical dynamics, including many applications of sea ice melting and freezing [28], continuous casting of steel [20], and lithium-ion batteries [15]. These therefore give rise to related important control and estimation problems, i.e., the boundary control and state observation of parabolic PDEs in [5]–[8], [11], [14], [18], [19], and [2], [22], [23] respectively.

B. Control of Parabolic PDE-ODE Systems

In addition to the aforementioned works on parabolic PDEs, topics concerning parabolic PDE-ODE coupled systems are also popular, which have rich physical background, such as coupled electromagnetic, coupled mechanical, and coupled chemical reactions [25]. Using the backstepping method, state-feedback and output-feedback control designs of a class of heat PDE-ODE coupled systems were presented in [24]–[26]. The problem of state observation is addressed for some parabolic PDE-ODE models in [1] and [27]. The sliding model control was proposed to achieve boundary feedback stabilization of a heat PDE-ODE cascade system with external disturbances in [29].

C. Control of ODE-PDE-ODE Systems

All aforementioned works consider actuation of PDE boundaries and ignore the dynamics of the actuator. However, sometimes the actuator dynamics may not be neglected, especially when dominant time constants of the actuator are closed to those of the plant. Considering the parabolic PDE-ODE coupled system with ODE actuator dynamics, it gives rise to a more challenging control/estimation problem of an ODE-PDE-ODE "sandwiched" system. Fewer attempts have been made on the boundary control of such an ODE-PDE-ODE system or PDE systems following ODE actuator dynamics where the controller acts. The boundary control of a viscous Burgers' equation with an integration at the input, which is regarded as a first-order linear ODE in the input channel, was considered in [17]. Backstepping state feedback control design for a transport PDE-ODE system where an integration at the input of the transport PDE was proposed in [9]. The control problem of an ODE with input delay and unmodeled bandwidth limiting actuator dynamics, which is represented by an ODE-transport PDE-ODE system where the input ODE is first order, is successfully addressed in [13]. Stabilization of 2×2 coupled linear first-order hyperbolic PDEs sandwiched around two ODEs was also achieved in [31].

In this paper, we use the combination of ODE backstepping and PDE backstepping methods to exponentially stabilize an ODE-heat PDE-ODE coupled system where the two ODEs are of arbitrary orders. An observer is designed to estimate all PDE and ODE states using only one PDE boundary value and then the observer-based output feedback law is proposed.

D. Main Contributions

1) This is the first result of stabilizing such an ODE-parabolic PDE-ODE "sandwiched" system where the control action only acts in one ODE.
2) Compared with our previous work [31] where a state-feedback control law was designed to stabilize the ODE-hyperbolic PDE-ODE system with the second-order input ODE and only a sketch of the design and analysis for an arbitrary-order input ODE was provided, in the present paper, we extend the hyperbolic PDE to a parabolic PDE, where challenges appear because of the higher-order spatial derivative and the inconformity between the orders of time and spatial derivatives. In addition, we design an observer to estimate all the states of the ODE-PDE-ODE system only using one PDE boundary value, and an output-feedback control law is proposed. Moreover, more detailed control design and stability analysis of the system where arbitrary-order ODEs sandwich around the PDE are presented.
3) Compared with the previous results about stabilizing ODE-transport PDE-ODE systems where both ODEs are first order [4], [9], [13], in addition to replacing the transport PDE by a heat PDE, we achieve a more challenging and general result where the orders of both ODEs sandwiching the PDE are arbitrary.

E. Organization

The rest of the paper is organized as follows. The concerned model is presented in Section II. The state-feedback control design combining the PDE backstepping and ODE backstepping is shown in Section III. The observer design and the output-feedback control law are proposed

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in Section IV. The simulation results are provided in Section V. The conclusion and future work are presented in Section VI.

Throughout this paper, the partial derivatives and total derivatives are denoted as $f_x(x,t) = \frac{\partial f}{\partial x}$, $f_t(x,t) = \frac{\partial f}{\partial t}$, $\partial_x^m f(x,t) = \frac{\partial^m f}{\partial x^m}$, $\partial_t^m f(x,t) = \frac{\partial^m f}{\partial t^m}$, $f'(x) = \frac{df}{dx}$, $\dot f(t) = \frac{df}{dt}$, $d_x^m f(x) = \frac{d^m f}{dx^m}$, $d_t^m f(t) = \frac{d^m f}{dt^m}$.

II. Problem Statement

We consider the following system where two ODEs sandwich around a heat PDE as:

$$\dot X(t) = AX(t) + Bu_x(0,t) \tag{1}$$

$$u_t(x,t) = qu_{xx}(x,t) \tag{2}$$

$$u(0,t) = C_X X(t) \tag{3}$$

$$u(1,t) = C_z Z(t) \tag{4}$$

$$\dot Z(t) = A_z Z(t) + B_z U(t) \tag{5}$$

$\forall (x,t) \in [0,1]\times[0,\infty)$, where $X(t) \in \mathbb{R}^{n\times 1}$, $Z(t) \in \mathbb{R}^{m\times 1}$ are ODE states, $n, m \in N^*$, N^* denoting positive integers. $u(x,t) \in R$ are states of the PDE. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ satisfy that the pair $[A,B]$ is controllable. $C_X \in \mathbb{R}^{1\times n}$ and $q \in R$ are arbitrary. $A_z \in \mathbb{R}^{m\times m}$ is

$$A_z = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ \bar a_1 & \bar a_2 & \bar a_3 & \cdots & \bar a_{m-1} & \bar a_m \end{bmatrix}_{m\times m} \tag{6}$$

where $\bar a_1, \ldots, \bar a_m$ are arbitrary constants. $B_z = [0, 0, \ldots, 1]^T \in \mathbb{R}^{m\times 1}$, $C_z = [1, 0, \ldots, 0] \in \mathbb{R}^{1\times m}$. Note that (A_z, B_z) and (A_z, C_z) are in the controllability normal form and observability normal form, respectively. $U(t)$ is the control input to be designed.

The control objective is to exponentially stabilize all ODE states $Z(t), X(t)$ and PDE states $u(x,t)$ by designing a control input $U(t)$ in one ODE, and control input itself should be guaranteed exponentially convergent as well.

The control design in this paper can be applied in the Stefan problem describing the melting or solidification mechanism with liquid-solid dynamics [16], i.e., heat PDE-ODE, driven by a thermal actuator described by an ODE at the boundary of the liquid phase.

III. State-Feedback Control Design

In this section, we combine the PDE backstepping (see Section III-A) and the ODE backstepping (see Section III-B) to design a state-feedback input. Exponential stability of the state-feedback closed-loop system is proved in Section III-C. The boundedness and exponential convergence of the state-feedback control input is proved in Section III-D.

A. Backstepping for PDE-ODE

We would like to use the infinite-dimensional backstepping transformations [12]

$$w(x,t) = u(x,t) - \int_0^x \phi(x,y)u(y,t)dy - \Phi(x)X(t) \tag{7}$$

with the inverse transformation as

$$u(x,t) = w(x,t) - \int_0^x \psi(x,y)w(y,t)dy - \Gamma(x)X(t) \tag{8}$$

to convert the original system (1)–(3) to

$$\dot X(t) = (A+BK)X(t) + Bw_x(0,t) \tag{9}$$

$$w_t(x,t) = qw_{xx}(x,t) \tag{10}$$

$$w(0,t) = 0 \tag{11}$$

where the right boundary condition $w(1,t)$ will be given later. $A+BK$ is Hurwitz by choosing the control parameter K since (A,B) is assumed controllable.

Mapping the original system (1)–(3) and the system (9)–(11) via the transformations (7), (8), the explicit solutions of kernels in (7) and (8) can be obtained as (12)–(16) in [30], which ensures the invertibility and boundedness of the backstepping transformation (7), (8). The detailed calculations of the kernels are shown in [25]. Note that dealing with the right boundary condition in the following steps will not affect determination of the kernels in (7) and (8).

Let us now calculate the right boundary condition $w(1,t)$ of (9)–(11). The right boundary condition $w(1,t)$ can be obtained by taking the m-order time derivative of the transformation (7) at $x=1$, inserting the original right boundary condition and the inverse transformation (8).

Considering (4) and (5) with (6), the right boundary condition of the original system can be written as

$$\begin{aligned}\partial_t^m u(1,t) &= \bar a_1 u(1,t) + \sum_{k=1}^{m-1} \bar a_{k+1}\partial_t^k u(1,t) + U(t)\\ &= \bar a_1 u(1,t) + \sum_{k=1}^{m-1} \bar a_{k+1} q^k \partial_x^{2k} u(1,t) + U(t).\end{aligned} \tag{12}$$

Taking the m times derivative in t of (7) at $x=1$, we have

$$\begin{aligned}\partial_t^m w(1,t) &= \partial_t^m u(1,t) - q^m \int_0^1 \partial_y^{2m}\phi(1,y)u(y,t)dy\\ &\quad + q^m \sum_{i=1}^{2m} (-1)^i \partial_y^{i-1}\phi(1,1)\partial_x^{2m-i}u(1,t)\\ &\quad - \Phi(1)A^m X(t) - \Bigg(q^m \sum_{i=1}^{2m}(-1)^i\partial_y^{i-1}\phi(1,0)\partial_x^{2m-i}\\ &\quad + \Phi(1)\sum_{i=1}^{m} A^{i-1}Bq^{m-i}\partial_x^{2(m-i)+1}\Bigg)u(0,t)\end{aligned} \tag{13}$$

for $m \in N^*$, where $\partial_x^k u(1,t) \triangleq \partial_x^k u(x,t)|_{x=1}$, $\partial_x^k u(0,t) \triangleq \partial_x^k u(x,t)|_{x=0}$. Insert (12) into (13) to replace $\partial_t^m u(1,t)$. Then rewrite u in (13) as w via the inverse transformation (8), where the k-order derivative of the inverse transformation (8) in x would be used

$$\begin{aligned}\partial_x^k u(x,t) &= \partial_x^k\left(w(x,t) - \int_0^x \psi(x,y)w(y,t)dy - \Gamma(x)X(t)\right)\\ &= \partial_x^k w(x,t) - \int_0^x \partial_x^k\psi(x,y)w(y,t)dy\\ &\quad - \sum_{j=0}^{k-1}\chi_{k\_j}(x)\partial_x^{k-j-1}w(x,t) - d_x^k\Gamma(x)X(t)\end{aligned} \tag{14}$$

for $k \in N^*$, where $\chi_{k\_j}(x)$ denoting the sum of j-order derivatives of $\psi(x,x)$ with respect to x, results from calculating $\partial_x^k(\int_0^x \psi(x,y)w(y,t)dy)$, as following:

$$\chi_{k\_j}(x) = \sum_{i=0}^{j} \bar\eta_{k\_j\_i}\partial_x^i\partial_y^{j-i}\psi(x,y)|_{(x,y)=(x,x)} \tag{15}$$

where constant coefficients $\bar{\eta}_{k\_j\_i}$ can be easily determined by calculating $\partial_x^k(\int_0^x \psi(x,y)w(y,t)dy)$ under some specific k according to the order of the plant.

After plugging (14) into (13) where $\partial_t^m u(1,t)$ has been replaced by (12), the right boundary condition of the system-w is obtained as

$$\begin{aligned}
&\partial_t^m w(1,t)\\
&= \Bigg[\bar{a}_1 + \sum_{k=1}^{m-1}\bar{a}_{k+1}q^k\partial_x^{2k} - \sum_{k=1}^{m-1}\bar{a}_{k+1}q^k\sum_{j=0}^{2k-1}\chi_{2k_j}(1)\partial_x^{2k-j-1}\\
&\quad + q^m\sum_{i=1}^{2m}(-1)^i\partial_y^{i-1}\phi(1,1)\partial_x^{2m-i} - q^m\sum_{i=1}^{2m}(-1)^i\partial_y^{i-1}\phi(1,1)\\
&\quad \times\sum_{j=0}^{2m-i-1}\chi_{2m-i_j}(x)\partial_x^{2m-i-j-1}\Bigg]w(1,t)\\
&\quad + \Bigg[-q^m\sum_{i=1}^{2m}(-1)^i\partial_y^{i-1}\phi(1,0)\partial_x^{2m-i}\\
&\quad + q^m\sum_{i=1}^{2m}(-1)^i\partial_y^{i-1}\phi(1,0)\sum_{j=0}^{2m-i-1}\chi_{2m-i_j}(0)\partial_x^{2m-i-j-1}\\
&\quad + \Phi(1)\sum_{i=1}^{m}A^{i-1}Bq^{m-i}\partial_x^{2(m-i)+1} - \Phi(1)\sum_{i=1}^{m}A^{i-1}Bq^{m-i}\\
&\quad \times\sum_{j=0}^{2(m-i)+1-1}\chi_{2(m-i)+1_j}(0)\partial_x^{2(m-i)+1-j-1}\Bigg]w(0,t)\\
&\quad - \int_0^1\Bigg[q^m\sum_{i=1}^{2m}(-1)^i\partial_y^{i-1}\phi(1,1)\partial_x^{2m-i}\psi(1,y) + \bar{a}_1\psi(1,y)\\
&\quad + \sum_{k=1}^{m-1}\bar{a}_{k+1}q^k\partial_x^{2k}\psi(1,y) + q^m\partial_y^{2m}\phi(1,y)\\
&\quad - q^m\int_y^1\partial_y^{2m}\phi(1,z)\psi(z,y)dz\Bigg]w(y,t)dy\\
&\quad + \Bigg[q^m\sum_{i=1}^{2m}(-1)^i\partial_y^{i-1}\phi(1,0)d_x^{2m-i}\Gamma(0) - \Phi(1)A^m\\
&\quad - \Phi(1)\sum_{i=1}^{m}A^{i-1}Bq^{m-i}d_x^{2(m-i)+1}\Gamma(0)\\
&\quad - q^m\sum_{i=1}^{2m}(-1)^i\partial_y^{i-1}\phi(1,1)d_x^{2m-i}\Gamma(1)\\
&\quad - \sum_{k=1}^{m-1}\bar{a}_{k+1}q^kd_x^{2k}\Gamma(1) - \bar{a}_1\Gamma(1)\\
&\quad + q^m\int_0^1\partial_y^{2m}\phi(1,y)\Gamma(y)dy\Bigg]X(t) + U(t) \qquad (16)
\end{aligned}$$

where some typical operators are

$$\partial_x^k w(1,t) \triangleq \partial_x^k w(x,t)|_{x=1},\ \partial_x^k w(0,t) \triangleq \partial_x^k w(x,t)|_{x=0}$$
$$\partial_y^k\phi(1,1) \triangleq \partial_y^k\phi(x,y)|_{(x,y)=(1,1)},\ d_x^k\Gamma(0) \triangleq d_x^k\Gamma(x)|_{x=0}.$$

More details of (16) please see (21) in [30]. Note that (16) is a m-order ODE system $w(1,t)$ with a number of PDE state perturbation terms. For clarity, (16) can be written as

$$\begin{aligned}
\partial_t^m w(1,t) &= \mathcal{B}w(1,t) + \mathcal{C}w(0,t)\\
&\quad - \int_0^1\mathcal{D}(y)w(y,t)dy + \mathcal{E}X(t) + U(t). \qquad (17)
\end{aligned}$$

Note that (17) is the right boundary condition of the system (9)–(11). $\mathcal{B},\mathcal{C},\mathcal{D}$, and $\mathcal{E}$ in (17) correspond to the parts including derivative operators in the square brackets before $w(1,t)$, $w(0,t)$, $w(y,t)$, $X(t)$ in (16). $\mathcal{D}(y)$ is a function of y and $\mathcal{E}$ is a constant vector. Note that $\mathcal{B}w(1,t) \triangleq (\mathcal{B}w(x,t))|_{x=1}$, $\mathcal{C}w(0,t) \triangleq (\mathcal{C}w(x,t))|_{x=0}$.

Theorem 1: Considering the system (1)–(6) and the backstepping transformation (7), (8), (16) holds for $m \in N^*$ which is the order of the ODE (5).

Proof: The proof is provided in the proof of Theorem 1 in [30]. ■

B. Backstepping for Input ODE With PDE State Perturbations

The following backstepping transformation [31] for the system-$\big(w(1,t), w_t(1,t), \ldots, \partial_t^{m-1}w(1,t)\big)$ (16) is made:

$$y_1(t) = w(1,t) \qquad (18)$$
$$y_2(t) = w_t(1,t) + \tau_1[w(1,t)] \qquad (19)$$
$$\cdots$$
$$\begin{aligned}
y_m(t) &= \partial_t^{m-1}w(1,t)\\
&\quad + \tau_{m-1}[w(1,t),\ldots,\partial_t^{m-2}w(1,t)] \qquad (20)
\end{aligned}$$

where $\tau_1,\ldots,\tau_{m-1}$ defined in the following steps are the virtual controls in the ODE backstepping method.

Step 1: We consider a Lyapunov function candidate as $V_{y1} = \frac{1}{2}y_1(t)^2$, taking the derivative of which we obtain $\dot{V}_{y1} = -c_1y_1(t)^2 + y_1(t)y_2(t)$ with the choice of $\tau_1 = c_1y_1(t)$, where c_1 is a positive constant to be determined later.

Step 2: A Lyapunov function candidate is considered as

$$V_{y2} = V_{y1} + \frac{1}{2}y_2(t)^2 = \frac{1}{2}y_1(t)^2 + \frac{1}{2}y_2(t)^2. \qquad (21)$$

Taking the derivative of (21), we have

$$\dot{V}_{y2} = -c_1y_1(t)^2 + y_1(t)y_2(t) + y_2(t)(y_3(t) - \tau_2 + \dot{\tau}_1).$$

Choosing $\tau_2 = \dot{\tau}_1 + y_1(t) + c_2y_2(t)$, we have

$$\dot{V}_{y2} = -c_1y_1(t)^2 - c_2y_2(t)^2 + y_2(t)y_3(t). \qquad (22)$$

Step 3: Similarly, a Lyapunov function candidate is considered as

$$\begin{aligned}
V_{ym} &= V_{y_{m-1}} + \frac{1}{2}y_m(t)^2 = \frac{1}{2}y_1(t)^2 + \frac{1}{2}y_2(t)^2\\
&\quad + \cdots + \frac{1}{2}y_{m-1}(t)^2 + \frac{1}{2}y_m(t)^2. \qquad (23)
\end{aligned}$$

Taking the derivative of (23), we have

$$\begin{aligned}
\dot{V}_{ym} &= -c_1y_1(t)^2 - c_2y_2(t)^2 - \cdots - c_{m-1}y_{m-1}(t)^2\\
&\quad + y_{m-1}(t)y_m(t) + y_m(t)\dot{y}_m(t). \qquad (24)
\end{aligned}$$

Considering (17) and (20), (24) can be rewritten as

$$\begin{aligned}
\dot{V}_{ym} &= -c_1y_1(t)^2 - c_2y_2(t)^2 - \cdots - c_{m-1}y_{m-1}(t)^2\\
&\quad + y_{m-1}(t)y_m(t) + y_m(t)\Big(U(t) + \mathcal{B}w(1,t)\\
&\quad + \mathcal{C}w(0,t) - \int_0^1\mathcal{D}(y)w(y,t)dy + \mathcal{E}X(t) + \dot{\tau}_{m-1}\Big) \qquad (25)
\end{aligned}$$

where

$$\tau_{m-1} = c_1 y_1^{m-2}(t) + y_1^{m-3}(t) + c_2 y_2^{m-3}(t) + y_2^{m-4}(t) + \cdots + c_{m-1} y_{m-1}(t),\ \forall m \geq 4. \tag{26}$$

Note $y_i^n(t)$ denotes n-order derivative of $y_i(t)$, $\forall i = 1, \ldots, m$.

Design the control input as

$$U(t) = -\mathcal{B}w(1,t) - \mathcal{C}w(0,t) - y_{m-1}(t) - \dot{\tau}_{m-1} - c_m y_m(t). \tag{27}$$

Recalling (18)–(20) and (26), we know

$$y_{m-1}(t) + \dot{\tau}_{m-1} = \sum_{i=0}^{m-1} \alpha_i(c_1, \ldots, c_{m-1})\partial_t^i w(1,t) \tag{28}$$

$$c_m y_m(t) = c_m \sum_{i=0}^{m-1} \beta_i(c_1, \ldots, c_{m-1})\partial_t^i w(1,t) \tag{29}$$

where α_i, β_i are constants depending on $c_1, \ldots, c_{m-1}$. The control law (27) then can be expressed as

$$U(t) = \mathcal{L}w(1,t) - \mathcal{C}w(0,t) \tag{30}$$

where $\mathcal{L} = -\mathcal{B} - \sum_{i=0}^{m-1} q^i(\alpha_i + c_m\beta_i)\partial_x^{2i}$.

Submitting (27) into (25), we get

$$\dot{V}_{ym} = -c_1 y_1(t)^2 - c_2 y_2(t)^2 - \cdots - c_m y_m(t)^2 + y_m(t)\left(-\int_0^1 \mathcal{D}(y)w(y,t)dy + \mathcal{E}X(t)\right) \tag{31}$$

where $c_1, \ldots, c_m$ are positive constants to be determined later.

C. Control Law and Stability Analysis

Substituting the backstepping transformation (7) into (30), we get the control input expressed by the original states

$$U(t) = \mathcal{L}u(1,t) - (\mathcal{L}\Phi(1) - \mathcal{C}\Phi(0))X(t) - \mathcal{C}u(0,t) - \mathcal{L}\int_0^1 \phi(1,y)u(y,t)dy + \bar{F}\left(u(0,t), \ldots, \partial_x^{2m-2}u(0,t)\right) \tag{32}$$

where the function $\bar{F}$ is obtained from

$$\bar{F} = \left(\mathcal{C}\int_0^x \phi(x,y)u(y,t)dy\right)\Big|_{x=0} \tag{33}$$

with $\mathcal{C}$ including differential operators $\sum_{i=0}^{2m-1}\partial_x^i$ defined before. The pending control parameters $c_1, \ldots, c_m$ included in $\mathcal{L}$ will be determined in the following stability analysis. According to the operators $\mathcal{L}$, $\mathcal{C}$, we know the signals used in the control law (32), (33) are $\sum_{i=0}^{2m-1}\partial_x^i u(1,t)$, $\sum_{i=0}^{2m-1}\partial_x^i u(0,t)$, $X(t)$, and $u(x,t)$. In order to ensure the control law is sufficiently regular, we will require the initial value $u(x,0)$ to be in $H^{2m}(0,1)$, which is defined as $H^{2m}(0,1) = \{u | u \in L^2(0,1), u_x \in L^2(0,1), \ldots, \partial_x^{2m-1}u \in L^2(0,1), \partial_x^{2m}u \in L^2(0,1)\}$ for $m \geq 1$, where $L^2(0,1)$ is the usual Hilbert space.

Theorem 2: Consider the closed-loop system consisting of the plant (1)–(5) and the control input (32), (33) with some control parameters $c_1, \ldots, c_m$, and initial values $u(x,0) \in H^{2m}(0,1)$. There exist constants $\Upsilon_s > 0$, $\lambda_s > 0$ such that

$$\Theta(t) \leq \Upsilon_s \Theta(0) e^{-\lambda_s t} \tag{34}$$

where $\Theta(t) = \left(\|u(\cdot,t)\|^2 + \|u_x(\cdot,t)\|^2 + |Z(t)|^2 + |X(t)|^2\right)^{\frac{1}{2}}$. $\|\cdot\|$ denotes the norm on $L^2(0,1)$, i.e., $\|u\| = \sqrt{\int_0^1 u(x,t)^2 dx}$ and $|\cdot|$ denotes the Euclidean norm.

Proof: We start from studying the stability of the target system. The equivalent stability property between the target system and the original system is ensured due to the invertibility of the PDE backstepping transformation (7) and the ODE backstepping transformation (18)–(20).

First, we study the stability of the PDE-ODE subsystem in the target system via Lyapunov analysis. Second, considering the Lyapunov analysis of the input ODE in Section III-B, Lyapunov analysis of the overall ODE-PDE-ODE system is provided, where the control parameters $c_1, c_2, \ldots, c_m$ in the control law (32) are determined.

1) Lyapunov Analysis for the PDE-ODE System: Defining

$$\Omega_0(t) = \|w(\cdot,t)\|^2 + \|w_x(\cdot,t)\|^2 + |X(t)|^2 \tag{35}$$

consider now a Lyapunov function

$$V_1(t) = X^T(t)PX(t) + \frac{a_0}{2}\|w(\cdot,t)\|^2 + \frac{a_1}{2}\|w_x(\cdot,t)\|^2 \tag{36}$$

where the matrix $P = P^T > 0$ is the solution to the Lyapunov equation $P(A+BK) + (A+BK)^T P = -Q$, for some $Q = Q^T > 0$. The positive parameters a_0, a_1 are to be chosen later.

From (35), we have $\theta_{01}\Omega_0(t) \leq V_1(t) \leq \theta_{02}\Omega_0(t)$ where θ_{01}, θ_{02} are positive constants. Applying Agmon's inequality, Young's inequality, and Cauchy–Schwarz inequality, taking the derivative of $V_1(t)$ along the trajectories of (9)–(11), we have

$$\begin{aligned}\dot{V}_1(t) \leq &-\left(\frac{a_1 q}{2} - \frac{4|PB|^2}{\lambda_{\min}(Q)} - \left(\frac{1}{4r_0}a_0 q + \frac{1}{4r_1}a_1\right)\right)w_x(0,t)^2\\ &-\left((a_0 - a_1)q - \left(\frac{1}{4r_0}a_0 q + \frac{1}{4r_1}a_1\right)\right)\|w_x\|^2\\ &-\left(\frac{1}{2}a_1 q - \left(\frac{1}{4r_0}a_0 q + \frac{1}{4r_1}a_1\right)\right)\|w_{xx}\|^2\\ &-\frac{3}{4}\lambda_{\min}(Q)|X(t)|^2 + r_0 a_0 q w(1,t)^2 + r_1 a_1 w_t(1,t)^2\end{aligned} \tag{37}$$

where $-\|w_{xx}\|^2 \leq 2\|w_x\|^2 - w_x(0,t)^2$ obtained from Agmon's inequality [24], [25] is used. Choosing parameters a_0, a_1 to satisfy

$$a_1 > \frac{8|PB|^2}{q\lambda_{\min}(Q)},\ a_0 > a_1 \tag{38}$$

with sufficiently large r_0, r_1 in (37), using Poincaré inequality, we obtain

$$\begin{aligned}\dot{V}_1(t) \leq &-\frac{1}{5}\left((a_0 - a_1)q - \left(\frac{1}{4r_0}a_0 q + \frac{1}{4r_1}a_1\right)\right)\|w_x\|^2\\ &-\frac{4}{5}\left((a_0 - a_1)q - \left(\frac{1}{4r_0}a_0 q + \frac{1}{4r_1}a_1\right)\right)\|w_x\|^2\\ &-\frac{3}{4}\lambda_{\min}(Q)|X(t)|^2 - \bar{\xi}_a w_x(0,t)^2\\ &+ r_0 a_0 q w(1,t)^2 + r_1 a_1 w_t(1,t)^2\\ \leq &-\frac{1}{5}\left((a_0 - a_1)q - \left(\frac{1}{4r_0}a_0 q + \frac{1}{4r_1}a_1\right)\right)(\|w_x\|^2 + \|w\|^2)\\ &-\frac{3}{4}\lambda_{\min}(Q)|X(t)|^2 - \bar{\xi}_a w_x(0,t)^2\\ &+ r_0 a_0 q w(1,t)^2 + r_1 a_1 w_t(1,t)^2\\ \leq &-\lambda_1 V_1(t) - \bar{\xi}_a w_x(0,t)^2 + r_0 a_0 q w(1,t)^2 + r_1 a_1 w_t(1,t)^2\end{aligned} \tag{39}$$

for some positive λ_1 and $\bar{\xi}_a$.

2) Lyapunov Analysis for the Overall ODE-PDE-ODE System: Recalling (36), and define a Lyapunov function

$$V(t) = V_1(t) + R_y V_{ym}(t) \tag{40}$$

where $R_y > 0$ is to be determined later. Defining

$$\Omega(t) = \|w(\cdot,t)\|^2 + \|w_x(\cdot,t)\|^2 + |X(t)|^2 + y_1(t)^2 + \cdots + y_m(t)^2 \tag{41}$$

we have

$$\theta_1 \Omega(t) \le V(t) \le \theta_2 \Omega(t) \tag{42}$$

with positive constants θ_1, θ_2. Taking the derivative of (40) and recalling (31) and (39), we get

$$\begin{aligned}\dot{V} \le& -\lambda_1 V_1(t) - \bar{\xi}_a w_x(0,t)^2 + r_0 a_0 q w(1,t)^2 + r_1 a_1 w_t(1,t)^2 \\ &- R_y c_1 y_1(t)^2 - R_y c_2 y_2(t)^2 - \cdots - R_y c_m y_m(t)^2 \\ &+ R_y y_m(t)\left(-\int_0^1 \mathcal{D}(x) w(x,t)dx + \mathcal{E}X(t)\right).\end{aligned} \tag{43}$$

Using (18) and (19) to replace $w(1,t)^2$, $w_t(1,t)^2$ in (43) as $y_1(t)^2, y_2(t)^2$, applying Young's inequality, Cauchy–Schwarz inequality, we have

$$\begin{aligned}\dot{V} \le& -\frac{\lambda_1}{2} V_1(t) - \left(\frac{1}{2}\lambda_1\theta_{01} - R_y \bar{r}_3 |\mathcal{E}|^2\right)|X(t)|^2 \\ &- \left(\frac{1}{2}\lambda_1\theta_{01} - R_y \bar{r}_4 \max_{0\le x\le 1}\{|\mathcal{D}(x)|\}\right)\int_0^1 w(x,t)^2 dx \\ &- \left(R_y c_1 - 2r_1 a_1 c_1^2 - r_0 a_0 q\right) y_1(t)^2 - \left(R_y c_2 - 2r_1 a_1\right) y_2(t)^2 \\ &- R_y c_3 y_3(t)^2 - \cdots - R_y c_{m-1} y_{m-1}(t)^2 \\ &- R_y\left(c_m - \frac{1}{4\bar{r}_3} - \frac{1}{4\bar{r}_4}\right) y_m(t)^2 - \bar{\xi}_a w_x(0,t)^2.\end{aligned} \tag{44}$$

Positive constants $\bar{r}_3, \bar{r}_4$ should satisfy

$$\bar{r}_3 < \frac{\lambda_1\theta_{01}}{2R_y|\mathcal{E}|^2},\ \bar{r}_4 < \frac{\lambda_1\theta_{01}}{2R_y \max_{0\le x\le 1}\{|\mathcal{D}(x)|\}}. \tag{45}$$

Choose the control parameter c_m in the control law (32) as

$$c_m > \frac{1}{4\bar{r}_3} + \frac{1}{4\bar{r}_4} \tag{46}$$

for $m > 2$ where R_y should be chosen as

$$R_y > \max\left\{\frac{2r_1 a_1 c_1^2 + r_0 a_0 q}{c_1}, \frac{2r_1 a_1}{c_2}\right\}. \tag{47}$$

$c_1, \ldots, c_{m-1}$ can be arbitrary positive constants. If $m = 2$, c_m would be chosen as $c_m > \frac{2r_1 a_1}{R_y} + \frac{1}{4\bar{r}_3} + \frac{1}{4\bar{r}_4}$ for $m = 2$, with choosing $R_y > \max\left\{\frac{2r_1 a_1 c_1^2 + r_0 a_0 q}{c_1}\right\}$.

We thus achieve

$$\dot{V} \le -\lambda V - \bar{\xi}_a w_x(0,t)^2 \le -\lambda V \tag{48}$$

for some positive λ.

Note that $m \ge 2$ in the above-mentioned proof because $w_t(1,t)^2$ is represented by $y_1(t)^2, y_2(t)^2$ in (44). If $m = 1$, as (67)-(68) in [30], $w_t(1,t)^2$ can be represented by $\|w\|^2 + |X(t)|^2 + w(1,t)^2$ where $|X(t)|^2, \|w\|^2$ can be "incorporated" by $-|X(t)|^2, -\|w\|^2$ in $\dot{V}_1$, and $w(1,t)^2$ can be "incorporated" by $-y_1(t)^2$ in $\dot{V}_{ym}$ with large enough c_1, and then (48) is also obtained.

From (41), (42), and (48), using the invertibility between $(y_1(t), \ldots, y_m(t))$ and $(w(1,t), w_t(1,t), \ldots, \partial_t^{m-1} w(1,t))$ via the backstepping transformation (18)–(20), and the invertibility between the target system $(w(x,t), X(t))$ and the original system $(u(x,t), X(t))$ via the backstepping transformation (7), recalling (4)–(6), we can conclude that the $(u(x,t), X(t), Z(t))$ system is exponentially stable in the sense of (34).

The proof of Theorem 2 is completed. ■

D. Boundedness and Exponential Convergence of the Control Input $U(t)$

In the last subsections, we have proposed the state-feedback control law and proved that all PDE and ODE states are exponentially stable in the origin in the state-feedback closed-loop system. In this subsection, we prove the exponential convergence and boundedness of the control input $U(t)$ (32) in the closed-loop system.

Theorem 3: In the closed-loop system including the plant (1)–(5) and the control input $U(t)$ (32), $|U(t)|$ is bounded by Υ_{sf} and is exponentially convergent to zero in the sense of $|U(t)| \le \Upsilon_{sf} e^{-\lambda_{sf} t}$ with the positive constants λ_{sf} and Υ_{sf} which only depends on initial values of the system.

Before the proof of Theorem 3, we present a lemma first. To investigate the boundedness and exponential convergence of the control input (32) where the highest-order derivative terms are $\partial_x^{2m-1} u(0,t)$, $\partial_x^{2m-1} u(1,t)$, we estimate the L_2 norm of the states up to $2m$-order spatial derivatives $\partial_x^{2m} u(x,t)$ in the following lemma.

Lemma 1: Consider the closed-loop system consisting of the plant (1)–(5) and the control input (32), (33) with some control parameters $c_1, \ldots, c_m$, and initial values $u(x,0) \in H^{2m}(0,1)$. Then, there exist constants $\Upsilon_{2m} > 0$ and $\lambda_{2m} > 0$ such that $\sum_{i=2}^{2m} \|\partial_x^i u(\cdot,t)\| \le \Upsilon_{2m} e^{-\lambda_{2m} t}$ where Υ_{2m} only depends on initial values.

Proof: The proof is provided in the proof of Lemma 1 in [30]. ■

Proof of Theorem 3: Recalling Theorem 2 and Lemma 1, we have the exponential stability estimates in the sense of the norm $\sum_{i=0}^{2m} \|\partial_x^i u(\cdot,t)\|$. Using Sobolev inequality, we obtain the exponential stability estimate in the sense of the norm $\|u(\cdot,t)\|_{C_{2m-1}}$, which gives the boundedness and exponential convergence of $U(t)$ by recalling Theorem 2. The proof of Theorem 3 is completed. ■

Note that when we mention the exponential stability result/estimate in the sense of the norm $N_0(t)$, it means there exist positive constants $\bar{\Upsilon} > 0$ and $\bar{\lambda} > 0$ such that $N_0(t) \le \bar{\Upsilon} e^{-\bar{\lambda} t}$ where $\bar{\Upsilon}$ only depends on initial values.

Brief summary: The backstepping approach [10] has been verified as a useful and new method for boundary control of distributed parameter systems. In the proposed method, a PDE backstepping transformation (7) is used to convert the original system to the system $(w(x,t), X(t))$ (9)–(11) and (17), where the state matrix $A + BK$ in (9) is Hurwitz and the left boundary condition is $w(0,t) = 0$, which are "stable like", but the right boundary condition (17) being a m-order ODE $w(1,t)$ with a number of PDE state perturbations. In order to form an exponentially stable target system, an ODE backstepping transformation (18)–(20) is adopted to convert the ODE states $w(1,t), \ldots, \partial_t^{m-1} w(1,t)$ at the right boundary to $y_1(t), \ldots, y_m(t)$, to build an exponentially stable system $(w(x,t), X(t), y_1(t), \ldots, y_m(t))$ under some control parameters $c_1, \ldots, c_m$ determined by Lyapunov analysis. Through the PDE backstepping and ODE backstepping transformations, the target system and the control law are obtained.

Comparing with a more naive approach, which is to design an intermediate control law for the PDE-ODE system and the intermediate control law to act as a reference to be tracked by the input ODE dynamics with m-order relative degree, the merit of the proposed design is avoiding taking m times derivative of the "intermediate control law" and producing high-order time derivatives of the state in the control law, especially high-order time derivatives of boundary states.

IV. Output Feedback Control Design

In Section III, a state-feedback control law at the input ODE is designed to exponentially stabilize the original ODE-PDE-ODE "sandwiched" system. However, the designed state-feedback control law requires the distributed states $u(x,t)$, which are always difficult to obtain in practice. In this section, we propose an observer-based output-feedback control law, which requires only one boundary value as the measurement. An observer is designed to reconstruct the distributed states $u(x,t)$ and two ODE states $Z(t), X(t)$ using only one boundary measurement $u_x(0,t)$ in Section IV-A. The observer-based output feedback control law is proposed in Section IV-B.

A. Observer Design

Suppose only one boundary value $u_x(0,t)$ is available for measurement, an observer is designed to reconstruct the states $u(x,t), Z(t), X(t)$ in this section.

Consider the observer

$$\dot{\hat{X}}(t) = A\hat{X}(t) + Bu_x(0,t) + P_0(u_x(0,t) - \hat{u}_x(0,t)) \quad (49)$$

$$\hat{u}_t(x,t) = q\hat{u}_{xx}(x,t) + p_1(x)(u_x(0,t) - \hat{u}_x(0,t)) \quad (50)$$

$$\hat{u}(0,t) = C_X\hat{X}(t),\ \hat{u}(1,t) = C_z\hat{Z}(t) \quad (51)$$

$$\dot{\hat{Z}}(t) = A_z\hat{Z}(t) + B_zU(t) + P_2(u_x(0,t) - \hat{u}_x(0,t)) \quad (52)$$

where the constant vectors P_0, P_2, and the function $p_1(x)$ are to be determined. Define the observer error as

$$\left(\tilde{u}(x,t), \tilde{Z}(t), \tilde{X}(t)\right) = \left(u(x,t), Z(t), X(t)\right) - \left(\hat{u}(x,t), \hat{Z}(t), \hat{X}(t)\right). \quad (53)$$

From (1)–(5) and (49)–(52), the observer error system can be written as

$$\dot{\tilde{X}}(t) = A\tilde{X}(t) - P_0\tilde{u}_x(0,t) \quad (54)$$

$$\tilde{u}_t(x,t) = q\tilde{u}_{xx}(x,t) - p_1(x)\tilde{u}_x(0,t) \quad (55)$$

$$\tilde{u}(0,t) = C_X\tilde{X}(t),\ \tilde{u}(1,t) = C_z\tilde{Z}(t) \quad (56)$$

$$\dot{\tilde{Z}}(t) = A_z\tilde{Z}(t) - P_2\tilde{u}_x(0,t). \quad (57)$$

We propose a transformation

$$\tilde{w}(x,t) = \tilde{u}(x,t) + \vartheta(x)\tilde{Z}(t) + \theta(x)\tilde{X}(t) \quad (58)$$

where the row vectors $\vartheta(x)$ and $\theta(x)$ are to be determined, to convert the error system (54)–(57) to the target error system

$$\tilde{w}_t(x,t) = q\tilde{w}_{xx}(x,t) \quad (59)$$

$$\tilde{w}(0,t) = 0,\ \tilde{w}(1,t) = 0 \quad (60)$$

$$\begin{bmatrix}\dot{\tilde{Z}}(t)\\ \dot{\tilde{X}}(t)\end{bmatrix} = \left(\begin{bmatrix}A_z & 0\\ 0 & A\end{bmatrix} + \begin{bmatrix}P_2\\ P_0\end{bmatrix}\begin{bmatrix}\vartheta'(0)\\ \theta'(0)\end{bmatrix}^T\right)\begin{bmatrix}\tilde{Z}(t)\\ \tilde{X}(t)\end{bmatrix} - \begin{bmatrix}P_2\\ P_0\end{bmatrix}\tilde{w}_x(0,t). \quad (61)$$

By mapping (54)–(57) and (59)–(61), $\vartheta(x), \theta(x)$ should satisfy the following two ODEs:

$$\vartheta(x)A_z - q\vartheta''(x) = 0 \quad (62)$$

$$\vartheta(0) = 0, \vartheta(1) = -C_z \quad (63)$$

$$\theta(x)A - q\theta''(x) = 0 \quad (64)$$

$$\theta(0) = -C_X, \theta(1) = 0 \quad (65)$$

and $p_1(x)$ should be chosen as

$$p_1(x) = -\vartheta(x)P_2 - \theta(x)P_0. \quad (66)$$

Conditions (62), (64), and (66) come from achieving (59) via (58) from (54)–(57). Conditions (63) and (65) result from (60).

The solution to (62), (63) can be represented by

$$\vartheta(x) = [0, \vartheta'(0)]\, e^{Fx}\begin{bmatrix}I\\ 0\end{bmatrix} \quad (67)$$

with $F = [0, \frac{A_z}{q}; I, 0]$ and I being an identity matrix with the appropriate dimension. Especially, for $x = 1$, it holds that

$$\vartheta(1) = [0, \vartheta'(0)]\, e^{F}\begin{bmatrix}I\\ 0\end{bmatrix} = -C_z. \quad (68)$$

According to Lemma 1 in [25], when if the matrix A_z has no eigenvalues of the form $-\bar{k}^2\pi^2$ for $\bar{k} \in N$

$$G = [0, I]e^F[I, 0]^T \quad (69)$$

is a nonsingular matrix. We then have $\vartheta'(0) = -C_zG^{-1}$.

Therefore the solution (67) is

$$\vartheta(x) = \left[0, -C_zG^{-1}\right] e^{Fx}\begin{bmatrix}I\\ 0\end{bmatrix}.$$

Similarly, we can obtain the solution of (64) and (65) as

$$\theta(x) = \left[-C_X, C_X\,[\,I\ 0\,]\,e^{F_1}\begin{bmatrix}I\\ 0\end{bmatrix}G_1^{-1}\right] e^{F_1x}\begin{bmatrix}I\\ 0\end{bmatrix}$$

where $G_1 = [0, I]e^{F_1}[I, 0]^T$ and $F_1 = [0, \frac{A}{q}; I, 0]$.

Let P_0, P_2 to be chosen so that the matrix

$$\bar{A} = A_a + \begin{bmatrix}P_2\\ P_0\end{bmatrix}B_a \quad (70)$$

is Hurwitz, where

$$A_a = \begin{bmatrix}A_z & 0\\ 0 & A\end{bmatrix},\ B_a = \begin{bmatrix}\vartheta'(0)\\ \theta'(0)\end{bmatrix}^T$$

(A_a, B_a) being supposed observable.

Thus, all the quantities needed to implement the observer (49)–(52) are determined. We then give the following theorem, which means the observer can effectively track the actual states in the plant (1)–(5).

Theorem 4: Supposing that the matrices A, A_z have no eigenvalues of the form $-\bar{k}^2\pi^2$, for $\bar{k} \in N$, consider the observer error system (54)–(57) obtained from the observer (49)–(52) and the plant (1)–(5) with initial values $\hat{u}(x,0) \in H^{2m}(0,1)$ and $u(x,0) \in H^{2m}(0,1)$. Then, there exist constants $\Upsilon_e > 0$ and $\lambda_e > 0$ such that

$$\Theta_e(t) \le \Upsilon_e\left(\Theta_e(0)^2 + \tilde{u}_x(0,0)^2\right)^{\frac{1}{2}} e^{-\lambda_e t} \quad (71)$$

where $\Theta_e(t) = \left(\sum_{i=0}^{2m}\|\partial_x^i\tilde{u}(\cdot,t)\|^2 + \left|\tilde{Z}(t)\right|^2 + \left|\tilde{X}(t)\right|^2\right)^{\frac{1}{2}}$.

Proof: The proof is provided in the proof of Theorem 4 in [30]. ■

B. Observer-Based Output Feedback Control Law

Replacing the states $u(x,t), X(t)$ in (32) as $\hat{u}(x,t), \hat{X}(t)$ defined through the observer (49)–(52), we obtain the output feedback control law

$$U_{of}(t) = \mathcal{L}\hat{u}(1,t) - \mathcal{C}\hat{u}(0,t) - (\mathcal{L}\Phi(1) - \mathcal{C}\Phi(0))\,\hat{X}(t) - \mathcal{L}\int_0^1 \phi(1,y)\hat{u}(y,t)dy + \hat{F}\left(\hat{u}(0,t), \ldots, \partial_x^{2m-2}\hat{u}(0,t)\right) \quad (72)$$

where $\hat{F} = \left(\mathcal{C}\int_0^x \phi(x,y)\hat{u}(y,t)dy\right)|_{x=0}$.

Under the proposed output feedback control law, the closed-loop system, which is shown in Fig. 1, is built. The exponential stability result of the closed-loop system

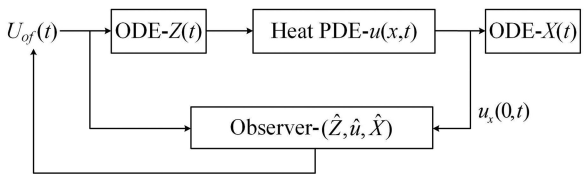

Fig. 1. Output-feedback closed-loop system consisting of the plant (1)–(5), observer (49)–(52), and control input (72).

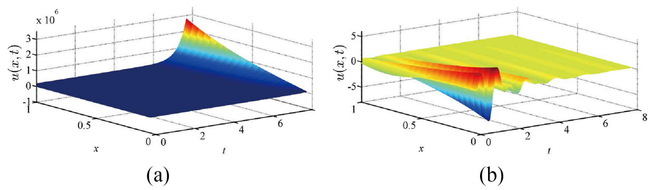

Fig. 2. Responses of the heat PDE states $u(x,t)$. (a) Uncontrolled case. (b) Controlled case.

in the sense of $\left(\|u(\cdot,t)\|^2 + \|u_x(\cdot,t)\|^2 + |X(t)|^2 + |Z(t)|^2 + \|\hat{u}(\cdot,t)\|^2 + \|\hat{u}_x(\cdot,t)\|^2 + \left|\hat{X}(t)\right|^2 + \left|\hat{Z}(t)\right|^2\right)^{\frac{1}{2}}$, and exponential convergence and boundedness of $U_{of}(t)$ are obtained. Please see Theorem 5 in [30].

V. Simulation

Consider the simulation example where the plant coefficients in (1)–(5) are $A = [1,1;1,0.5]$, $A_z = [0,1;1,1]$, $B_z = B = [0,1]^T$, $C_X = C_z = [1,0]^T$, and $q = 1$. Two ODEs sandwiching the heat PDE are considered as two-order systems here, i.e., $m = 2$ in the above-mentioned design and analysis, because the second-order ODE is a classic system which can describe many actuator and sensor dynamics. The simulation is conducted based on the finite difference method with dividing the spatial and time domains into a grid as $x_0,\ldots,x_f$ and $t_0,\ldots,t_{n^*}$, respectively, where the time step and space step sizes are 0.001 and 0.05. The initial conditions of the plant are defined as $u(x,0) = \sin(2\pi x)$, $X(0) = [x_1(0), x_2(0)]^T = [u(0,0),0]^T$, $Z(0) = [z_1(0), z_2(0)]^T = [u(1,0),0]^T$. The initial conditions of the observer (49)–(52) are $\hat{u}(x,0) = 0$, $\hat{X}(0) = \hat{Z}(0) = [0,0]^T$. Choose the control parameters $c_1 = c_2 = 3$, $K = [-10,-5]$, $P_0 = [-2,-4]^T$, and $P_2 = [-4,-12]^T$. Apply the output feedback control law (72) with $m = 2$, which is constructed by the states $\sum_{i=0}^{3}\partial_x^i\hat{u}(1,t)$, $\sum_{i=0}^{3}\partial_x^i\hat{u}(0,t)$, $\hat{X}(t)$, and $\hat{u}(x,t)$ of the observer (49)–(52) built using the measurement $u_x(0,t)$, into the plant (1)–(5). Note that the third-order spatial derivatives are determined by finite difference method as

$$\hat{u}_{xxx}(1,t_j) = \frac{\hat{u}(x_f,t_j) - 3\hat{u}(x_{f-1},t_j) + 3\hat{u}(x_{f-2},t_j) - \hat{u}(x_{f-3},t_j)}{\Delta h^3}$$

where Δh is spatial step size and t_j is the current time point. $\hat{u}_{xxx}(1,t_j)$ is used to determine $U_{of}(t_{j+1})$.

The responses of the output-feedback closed-loop system are shown in the following. As Fig. 2 shows, the response $u(x,t)$ of the heat PDE exhibits unstable behavior in the uncontrolled case while the convergent manner of the response $u(x,t)$ is achieved when we apply the proposed output feedback control law (72). Similarly, Figs. 3 and 4 show that the ODE states $X(t) = [x_1(t), x_2(t)]^T$ and $Z(t) = [z_1(t), z_2(t)]^T$ are also convergent to zero in the output-feedback closed-loop system. It can be seen in Fig. 5(a) that the observer errors $\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t)$ also converge fast to zero in the closed-loop system. Moreover, Fig. 5(b) shows that the output-feedback control input are bounded and convergent to zero.

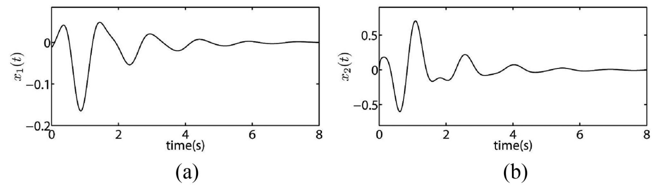

Fig. 3. Responses of the ODE state $X(t)$ under the output-feedback control law (72). (a) $x_1(t)$. (b) $x_2(t)$.

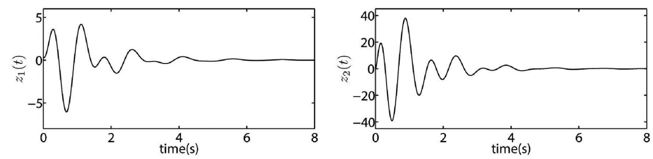

Fig. 4. Responses of the ODE state $Z(t)$ under the output feedback control law (72). (a) $z_1(t)$. (b) $z_2(t)$.

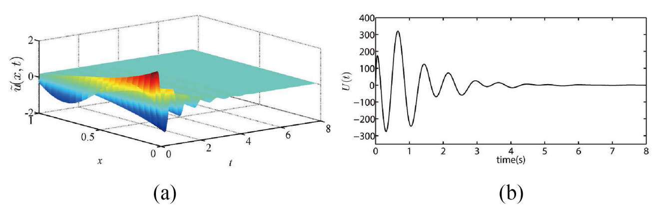

Fig. 5. (a) Observer errors $\tilde{u} = u - \hat{u}$ of the observer (49)–(52). (b) Output feedback control law (72).

VI. Conclusion and Future Work

In this paper, we present a methodology combining PDE backstepping and ODE backstepping to stabilize a parabolic PDE sandwiched between two arbitrary-order ODEs. An observer is also designed only using one PDE boundary value $u_x(0,t)$ to reconstruct all PDE and ODE states. The observer-based output-feedback control law is proposed and the exponential stability of the closed-loop system is proved via Lyapunov analysis. Moreover, the boundedness and exponential convergence of the designed control input is also proved in this paper. These theoretical results are verified via the simulation as well. In the future work, more general ODE dynamics in the input channel will be considered in the control design.

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